Simultaneous Congruence of Convex Compact Sets of Hermitian Matrices with Constant Rank

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ABSTRACT

Let S be a compact convex set of $n\times n$ hermitian matrices ($n\geqslant 2$). Suppose every member of S is nonsingular and has exactly one negative eigenvalue. Let $(\varepsilon_1,\dots,\varepsilon_n)$ be any ordered n-tuple from the set $\{-1,1\}$. One of our main results is that a nonsingular matrix X exists such that, for every A in S and every $1\leqslant j\leqslant n$, the (j,j) entry of X^*AX has sign ε_j. A similar result, with only negative ε_j allowed, is proved also for a compact convex set S of $n\times n$ hermitian matrices such that every member of S has the same rank and exactly one negative eigenvalue.

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I. INTRODUCTION

Let S be a set of $n \times n$ complex hermitian matrices, X an $n \times n$ nonsingular complex matrix, and T the set of matrices X^*AX for which A is in S. Then we say the matrices of T are *simultaneously congruent* to those of S. (We shall use "congruent" in the sense of "hermitian-congruent" or "conjunctive" in this paper.) The problem of finding a form to which the matrices of S can be brought by a simultaneous congruence is important and has been studied for a long time. For a singleton set $S = \{A\}$ Sylvester's law provides a complete solution in terms of the ordered triple $i(A) = (r, s, t)$, where A has r positive, s negative, and t zero eigenvalues $(r + s + t = n)$. We refer to $i(A)$ as the *inertia* of A. For simultaneous congruence of ordered pairs, results go back a century or so and are discussed, for instance, in [7], [8], and [2]. When S has more than two members, one must make some additional assumptions to achieve useful results. As an example, if the matrices $A_1, \dots, A_m$ commute pairwise, they can be diagonalized by a simultaneous unitary congruence.

In this paper we shall study simultaneous congruence of convex compact sets S of hermitian matrices under the assumption that all matrices of S have the same rank and have exactly one negative eigenvalue (or exactly one positive eigenvalue). It is easy to see that the assumption that each matrix in S has the same rank is equivalent to the formally stronger assumption that each has the same inertia (since S is convex).

Let us give an example of such a situation. Suppose S is a set of $n \times n$ hermitian matrices such that the following is true of each matrix A in S:

(i) the leading $r \times r$ principal submatrix of A is positive definite;
(ii) the succeeding $s \times s$ principal submatrix of A is negative definite;
(iii) the last $n - r - s$ rows and columns of A are zero.

Then we say the set S is *inertia explicit*. We note that if S is inertia explicit, then so is the nonzero part of the cone generated by S (i.e., the set of all nonzero nonnegative linear combinations of the matrices of S). In particular, if $\{A_1, \dots, A_m\}$ is inertia explicit, then so is the *convex hull*

$$\mathrm{Co}(A_1, \dots, A_m) = \left\{ \sum_{i=1}^{m} \lambda_i A_i : \sum_{i=1}^{m} \lambda_i = 1 \text{ and each } \lambda_i \geqslant 0 \right\}$$

of $A_1, \dots, A_m$.

Now suppose a set S is simultaneously congruent to an inertia explicit set, i.e., for some nonsingular matrix X the set $\{X^*AX : A \in S\}$ is inertia explicit. Then we say that S is *simultaneously inertia explicable*. In this case also, S has constant inertia and so has the convex hull of S.

It is natural to ask if the converse holds: is every convex set of constant inertia simultaneously inertia explicable? It was proved in [5] that the answer is no in general (even for compact sets) and is yes in the case of the convex hull of matrices $A_1, \dots, A_m$ where at least one of the following conditions holds:

(i) $m = 2$ and A_1 is nonsingular;
(ii) A_1 is semidefinite;
(iii) the matrices $A_1, \dots, A_m$ are 2×2 and real.

In view of this result, it is of interest to find a form to which compact convex sets S of hermitian matrices with constant rank can be brought by simultaneous congruence. In this paper we shall give a partial solution to this problem.

The main result (Theorem 1) states that by simultaneous congruence one can achieve any prescribed pattern of signs for the diagonal entries in each matrix A in S, provided that the matrices in S are nonsingular and have exactly one negative eigenvalue. If the assumption of nonsingularity of matrices in S is dropped, one can achieve by simultaneous congruence any prescribed pattern of negative signs and zeros for the diagonal entries, provided the number of negative signs is at least the common rank of matrices in S.

We shall also give a general construction (in Theorem 4) of examples showing the existence of real symmetric matrices A_1, A_2, A_3 which are not simultaneously inertia explicable but have constant rank on their convex hull.

Another motivation for this work comes from the study of a certain linear-preserver problem. Let $n \geqslant 2$, and fix an ordered triple (r, s, t) of integers $\geqslant 0$ with r and s positive and $r + s + t = n$. Let T be a linear transformation on the vector space of all $n \times n$ complex matrices, and let T satisfy the following: If A is hermitian and $i(A) = (r, s, t)$, then the same is true of $T(A)$. One is tempted to believe that T must be a congruence, possibly followed by a transposition, and, if $r = s$, possibly followed by negation. We believe the problem is open. Some limited results have been obtained in [3] and [4]. For a look at the definite case, see [1].

II. STATEMENT OF RESULTS

THEOREM 1.

(a) *Let S be a nonempty compact convex set of $n \times n$ nonsingular hermitian matrices ($n \geqslant 2$) having exactly one negative eigenvalue each, counting multiplicities. Let $(\varepsilon_1, \dots, \varepsilon_n)$ be any ordered n-tuple of signs ± 1.*

Then there is a nonsingular complex matrix X such that for each A in S and each $j \leqslant n$ the (j, j) entry of X^*AX has sign ε_j.

(b) *Let S be a nonempty compact convex set of $n \times n$ hermitian matrices such that each $A \in S$ has exactly one negative eigenvalue and the same rank p, where $p \leqslant n$. Then for any integer q, $p \leqslant q \leqslant n$, there is a nonsingular complex matrix X such that for each $A \in S$, each $j \leqslant q$, and each k and $l > q$ the (j, j) entry of X^*AX is negative and the (k, l) entry of X^*AX is zero.*

In both parts (a) *and* (b), *if all A in S are real, then X can be chosen real as well.*

The following open problem arises naturally in view of Theorem 1(a): do the conclusions of Theorem 1 still follow if one drops the requirement that each matrix in S be nonsingular? For $n = 3$ the answer is yes. More exactly, the following result holds.

THEOREM 2. *Let S be a nonempty compact convex set of $n \times n$ indefinite hermitian matrices such that each $A \in S$ has the same rank p. Assume $n \leqslant 3$. Let $(\varepsilon_1, \dots, \varepsilon_q)$ be any ordered q-tuple of signs ± 1 with $p \leqslant q \leqslant n$. Then there is a nonsingular complex matrix X such that, for each A in S each $j \leqslant q$ and each k and $l > q$, the (j, j) entry of X^*AX has sign ε_j and the (k, l) entry of X^*AX is zero. If all matrices in S are real, then X may be chosen real as well.*

It is an open problem whether Theorem 2 holds for $n \geqslant 4$.

REMARK 1. Theorem 1(b) implies that the matrices in S have a common $(n-p)$-dimensional totally isotropic subspace (i.e., such that $x^*Ax = 0$ for every x in this subspace and every A in S). In general, such a subspace is not unique. It will be seen from the proof of Theorem 1(b) that the kernel of any matrix in S is such a subspace. Note that the hypotheses of Theorem 1(b) do not imply that the kernels of the matrices in S coincide, as the following example shows:

$$S = \mathrm{Co}\{A_1, A_2\},$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

REMARK 2. Theorem 1 remains valid if the word "negative" is replaced by the word "positive." One proves this by applying Theorem 1 to the set $\{A : -A \in S\}$.

REMARK 3. Theorem 1 enables us to settle completely the problem of simultaneous inertia explicability in the 2×2 case for compact convex sets of matrices (or more generally for compact sets, since the convex hull of a compact set must be compact in a finite-dimensional space). In particular, a compact convex set S of 2×2 hermitian matrices is simultaneously inertia explicable if and only if all matrices in S have the same rank; moreover, if all matrices in S are real here, then the congruence which makes S simultaneously inertia explicable can be chosen real. Indeed, disregarding the trivial cases where S consists only of the zero matrix or only of positive definite or negative definite matrices, we have three cases left: (1) each matrix in S is indefinite; (2) each matrix in S is positive semidefinite of rank 1; and (3) each matrix in S is negative semidefinite of rank 1. The first case is covered by Theorem 1(a), and the second and third cases are taken care of by Lemma 6 below. In [5] these results were proved for the convex hull S of a finite set in the case where all matrices of S are real and nonsingular and in the case where all matrices are complex and singular.

The main part of Theorem 1 is included in the following theorem, which deserves to be stated separately.

THEOREM 3. *Let S be a nonempty compact convex set of $n\times n$ hermitian matrices of inertia $(n-1,1,0)$. Then*

(a) *there is an $n\times 1$ vector x such that $x^*Hx<0$ for every H in S; and*

(b) *if $n\geqslant 2$, there is an $n\times 1$ vector y such that $y^*Hy>0$ for every H in S.*

If all matrices in S are real, then x and y may be chosen real here.

Theorem 3 says that there is a vector x on which all the hermitian forms x^*Hx for H in S are negative (or all are positive). It may be that this is the best possible result, in the following sense: We conjecture that for each $n\geqslant 2$ there exist nonsingular indefinite hermitian $n\times n$ matrices $A_1,\dots,A_m$ with constant rank on their convex hull for which there is no 2-dimensional subspace on which all m hermitian forms x^*A_ix are positive definite or on which all are negative definite. For $n\leqslant 3$ this conjecture has been proved (see [5] or Theorem 4 below).

The following example shows that the compactness hypothesis in Theorem 1 cannot be dropped:

EXAMPLE. Let $S=\{A_m : m \text{ is real}\}$, where

$$A_m=\begin{bmatrix} m & 1\\ 1 & 0\end{bmatrix}.$$

Clearly S is convex (but not compact) and every matrix in S has inertia $(1,1,0)$; however, there is no vector x such that $x^*Ax > 0$ for every $m < 0$. Indeed, write $x = (u, v)^t$; then

$$x^*A_m x = m|u|^2 + u\bar{v} + \bar{u}v.$$

Suppose this were positive for all $m < 0$; then we would have $u \neq 0$, so we would have $|u|^2[m + 2\operatorname{Re}(v/u)] > 0$ for all $m < 0$, an obvious contradiction. Similarly, there is no vector y such that $y^*A_m y < 0$ for all $m > 0$.

THEOREM 4. *Let $n \geqslant 3$. Then we can construct three $n \times n$ real symmetric matrices whose convex hull has constant inertia $(n-1,1,0)$ but which are not simultaneously inertia explicable over the real field.*

We see from Theorem 4 that the hypotheses of Theorem 1(a), in the case that S is the convex hull of at least three matrices, are not sufficient to guarantee that S is simultaneously inertia explicable. This fact was already observed in [5], and for $n = 3$ Theorem 4 was demonstrated there. The proof of Theorem 4 given below will provide better insight into the structure of the three matrices than the example given in [5]. We should also note that we do not have a similar example for complex matrices. In other words, the matrices A_1, A_2, and A_3 given by Theorem 4 may well be simultaneously inertia explicable if we allow congruence by a complex nonsingular matrix.

The rest of this paper is devoted to the proofs of Theorems 1, 2, 3, and 4.

III. PROOF OF THEOREM 3

First note that the result is trivially true if $n = 1$. Thus let $p \geqslant 2$ and suppose (as our inductive assertion) that the theorem is true for $n = p - 1$. We shall prove that the theorem holds for $n = p$. Let V be the real vector space of $p \times p$ hermitian matrices, and P be the set of positive semidefinite hermitian matrices in V. Then P is a closed convex cone in V, and so is $-P$, the set of negative semidefinite matrices in V; moreover, $P \cap (-P) = \{0\}$ (i.e., P and $-P$ are "pointed" cones). Let G be the convex cone generated by S, that is,

$$G = \{kH : k \geqslant 0 \text{ and } H \in S\}.$$

It is easily seen that G is closed [and is pointed, i.e., $G \cap (-G) = \{0\}$], because S is compact (and $0 \notin S$). Since $G \cap P = \{0\}$ the separation theorem for closed cones (see, e.g., p. 96 in [6]) states that $G^* \cap (-P^*)$ has nonempty interior, where for a closed cone L in V we denote by L^* the closed cone dual to L, i.e., L^* is the set of all linear functionals f on V such that $f(x) \geqslant 0$ for all x in L. Furthermore, since $-P$ is not included in G, its dual cone $-P^*$ does not include G^*, so there exists a linear functional f on V which lies in the interior of G^* and on the boundary of $-P^*$. For such an f we have that $f(H) \geqslant 0$ for all H in $G \cup (-P)$. Moreover, $f(H) > 0$ for all nonzero H in G because f is in the interior of G^*, and $f(H) = 0$ for some nonzero H in $-P$ because f is on the boundary of $-P^*$.

Now, every linear functional f on V must have the form

$$f(H) = \text{trace}(-AH) \qquad \text{for all } H \text{ in } V$$

for some matrix A in V. Since our (abovementioned) f belongs to the boundary of $-P^*$, the matrix A here is positive semidefinite and singular. Since the positive semidefinite $p \times p$ matrices of rank $p-1$ are dense in the set of all singular positive semidefinite $p \times p$ matrices, we may (and shall) assume that A has rank $p-1$. Then there is a nonsingular $p \times p$ matrix C such that $A = C^*(I_{p-1} \oplus 0)C$. Let

$$T = CSC^* = \{CHC^* : H \in S\}$$

and

$$T' = \left\{ K_1 : K_1 \text{ is } (p-1)\times(p-1) \text{ and } \begin{bmatrix} K_1 & K_3 \\ K_3^* & K_2 \end{bmatrix} \in T \text{ for some } K_2, K_3 \right\}.$$

Then T and hence T' are compact convex sets of hermitian matrices, and each member of T has exactly one negative eigenvalue and $p-1$ positive eigenvalues. Thus, by the interlacing eigenvalue property for principal submatrices of hermitian matrices, each K_1 in T' has at least $p-2$ positive eigenvalues. However, each K_1 in T' has trace

$$\begin{aligned} \text{trace}\, K_1 &= \text{trace}\left[(I_{p-1} \oplus 0)K\right] = \text{trace}\left[(C^*)^{-1}AC^{-1}K\right] \\ &= \text{trace}\left[AC^{-1}K(C^*)^{-1}\right] \end{aligned}$$

for some K in T; hence trace $K_1 < 0$, since $C^{-1}K(C^{*-1}) \in S$ for every K in T. Thus each K_1 in T' has at least one negative eigenvalue and hence has exactly one negative eigenvalue. Thus by our induction assertion there is a $(p-1)\times 1$ vector x_1 such that $x_1^*K_1x_1 < 0$ for each K_1 in T', and if $p > 2$ there is a $(p-1)\times 1$ vector y_1 such that $y_1^*K_1y_1 > 0$ for each K_1 in T'. Put x and y equal to the respective $p\times 1$ vectors $[x_1^t\ \ 0]^t$ and $[y_1^t\ \ 0]^t$, and then put $u = C^*x$ and $v = C^*y$. Then for each H in S we have

$$u^*Hu = x^*CHC^*x = x_1^*K_1x_1 < 0$$

for some K_1 in T'. Likewise, $v^*Hv > 0$ for each H in S when $p > 2$. This completes the proof of the induction step, except for the (b) part when $p = 2$; however, this part follows by applying the (a) part when $p = 2$ to the set $-$S.

The arguments used to prove Theorem 3 can be used to prove also the following fact, which slightly generalizes part (a) of Theorem 3:

PROPOSITION 5. *Let S be a nonempty compact convex set of $n\times n$ hermitian matrices such that each matrix in S has one eigenvalue <0 and $n-1$ eigenvalues $\geqslant 0$. Then there exists an $n\times 1$ vector x (which can be taken real if all matrices of S are real) such that $x^*Ax < 0$ for all A in S.*

IV. PROOF OF THEOREMS 1 AND 2

We begin with the proof of part (a) of Theorem 1. Given $\varepsilon_1,\dots,\varepsilon_n$ as in Theorem 1(a), let $J_\pm = \{j: 1\leqslant j\leqslant n,\ \varepsilon_j = \pm 1\}$. By Theorem 3 there exists a linearly independent set $\{x_j\}$, $j\in J_-$, of vectors in $\mathbb{C}^n$ such that $x_j^*Hx_j < 0$ for all $H\in$ S. Again by Theorem 3 there is a set $\{x_j\}$, $j\in J_+$, such that $x_j^*Hx_j > 0$ for $H\in$ S and $j\in J_+$, and such that $x_1,\dots,x_n$ form a basis of $\mathbb{C}^n$. Put $X = [x_1,\dots,x_n]$ to satisfy the requirements of Theorem 1(a). If all $H\in$ S are real, choose real x_j by Theorem 3 again.

For the proof of Theorem 1(b) we need the following lemma.

LEMMA 6. *Let A_1 and A_2 be $n\times n$ hermitian matrices such that every matrix in their convex hull has the same rank. Then $x^*A_2x = 0$ for every x in the kernel of A_1.*

Proof. Since the hypothesis and conclusion are invariant under simultaneous congruence of A_1 and A_2, it suffices to prove Lemma 6 for the case where A_1 and A_2 have the form

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & A_{11} \end{bmatrix}, \qquad A_2 = \begin{bmatrix} A_{21} & A_{22}^* \\ A_{22} & A_{23} \end{bmatrix},$$

where A_{11} is a $p \times p$ nonsingular diagonal matrix and A_{21} is an $(n-p)\times(n-p)$ diagonal matrix. We need to prove that $A_{21} = 0$. The hypothesis of Lemma 6 implies that for all $\mu > 0$ the rank of $\mu A_1 + A_2$ is p. If we examine $(p+1)\times(p+1)$ subdeterminants of $\mu A_{11} + A_{23}$, we easily verify that $A_{21} = 0$. ■

Now assume the hypotheses of Theorem 1(b) are satisfied. Let A_1 be a fixed matrix of S, and then pick a linearly independent set of $n-q$ vectors $x_{q+1},\dots,x_n$ from the kernel of A_1. Then by Lemma 6, $x^*Hx = 0$ for every H in S and every x in $\operatorname{span}\{x_{q+1},\dots,x_n\}$.

We shall now show that the set Z_- is nonempty, where

$$Z_- = \{x : x^*Hx < 0 \text{ for all } H \text{ in } S\}.$$

Indeed, let ε be a positive number so small that every matrix of the set

$$S + \varepsilon I = \{H + \varepsilon I : H \in S\}$$

has a negative eigenvalue. (Such an ε exists because of the compactness of S and the continuity of the ordered eigenvalues of hermitian matrices.) It is easily seen that $S + \varepsilon I$ is compact and convex, and every matrix of $S + \varepsilon I$ has $n-1$ positive eigenvalues. By Theorem 3 there exists a vector x such that $x^*(H+\varepsilon I)x < 0$ for every $H \in S$. But then for every H in S we have

$$x^*Hx = x^*Kx - \varepsilon x^*x < 0$$

for some K in $S + \varepsilon I$. Thus Z_- is nonempty.

As Z_- is nonempty and open, there exists a set $\{x_1,\dots,x_q\}$ which belongs to Z_- and is such that the set $x_1,\dots,x_n$ is linearly independent. Put $X = [x_1,\dots,x_n]$ to satisfy the requirements of Theorem 1(b). If all $H \in S$ are real, then analogously one shows that x_j, in addition to the above properties, can be chosen real.

We pass now to the proof of Theorem 2. The case $p = n$ is taken care of by Theorem 1. The only remaining case to consider is $n = 3$, $p = 2$; then each member of S has one negative, one positive, and one zero eigenvalue. As in the proof of Theorem 1(b), one shows that the set

$$Z_- = \{x : x^*Hx < 0 \text{ for all } H \text{ in } S\}$$

is nonempty. Applying this observation to the set $\{-H : H \in S\}$, we obtain that

$$Z_+ = \{x : x^*Hx < 0 \text{ for all } H \text{ in } S\}$$

is also nonempty. Let $\varepsilon_1, \dots, \varepsilon_q$ be as in Theorem 2. If $q = 3$, use the nonemptiness and openness of $Z_\pm$ to choose linearly independent 3-dimensional vectors x_1, x_2, x_3 such that $x_j \in Z_{\varepsilon_j}$ for $j = 1,2,3$. If $q = 2$, then first pick a nonzero vector x_3 from the kernel of a fixed matrix in S and then choose x_1 and x_2 such that $x_j \in Z_{\varepsilon_j}$ for $j = 1,2$ and the set x_1, x_2, x_3 is linearly independent. In either case $X = [x_1, x_2, x_3]$ satisfies the requirements of Theorem 2.

V. PROOF OF THEOREM 4

Consider the hyperplane $N = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n = 1\}$. Let E_1 and E_2 be two ellipsoids in N with the following properties:

(i) the intersection of their interiors E_1^0 and E_2^0 is not empty;
(ii) E_1 and E_2 lie in the half space $x_1 \leqslant 0$;
(iii) the boundaries ∂E_1 and ∂E_2 of E_1 and E_2, respectively, are tangent to the hyperplane $H = \{x \in N : x_1 = 0\}$ at distinct points Q_1 and Q_2 respectively.

Next construct a hyperboloid of two sheets E_3 in N with the following properties:

(i) the axis of E_3 is the line $\{(x_1, 0, \dots, 0, 1) : x_1 \in \mathbb{R}\}$;
(ii) one sheet of E_3 is in the half space $x_1 < 0$ and meets $E_1^0 \cap E_2^0$;
(iii) the second sheet is in the half space $x_1 \geqslant 0$ and is tangent to the hyperplane H at a point Q_3 on the line segment Q_1Q_2 (see Figure 1).

Now we go projective. For each $v \in \mathbb{R}^n$, $v \neq 0$, the line spanned by v meets N in a unique point, possibly at infinity, which we call $\pi(v)$. So $\mathbb{P}^{n-1} = \pi(\mathbb{R}^n \setminus \{0\})$ is the $(n-1)$-dimensional real projective space.

Let A_1, A_2, A_3 be $n \times n$ real symmetric matrices such that the image of $\{x \in \mathbb{R}^n \setminus \{0\} : x^*A_ix \leqslant 0\}$ when projected by π into N is E_i, $i = 1,2,3$. Then each A_i has inertia $(n-1, 1, 0)$.

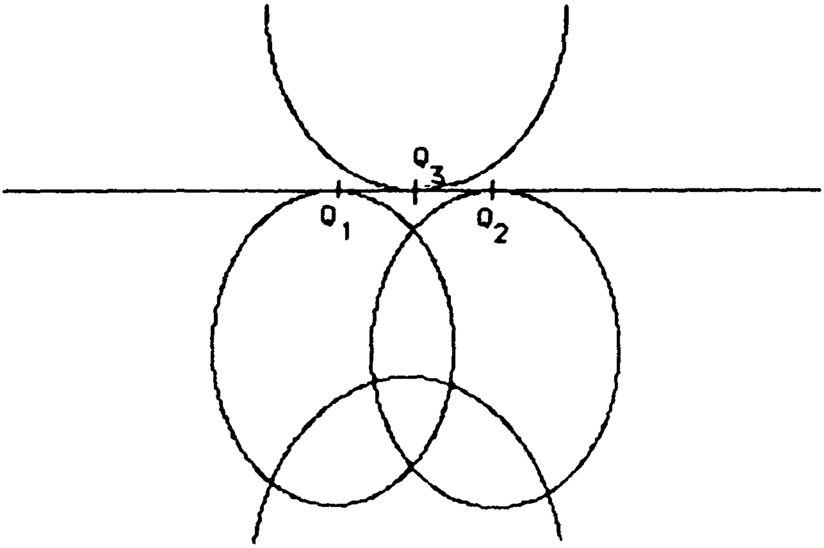

FIG. 1.

LEMMA 7. *With* A_1, A_2, A_3 *as above,* $\mathrm{Co}(A_1, A_2, A_3)$ *has constant inertia* $(n-1,1,0)$, *but* A_1, A_2, A_3 *are not simultaneously inertia explicable.*

Proof. Let $v_i \in \mathbb{R}^n \setminus \{0\}$ be such that $\pi(v_i) = Q_i$, $i = 1,2,3$. For a nonzero vector $u = (u_1, \dots, u_n)$ with $u_1 = 0$ we have $u^*A_1u \geqslant 0$, and the equality holds here if and only if $u \in \mathrm{Span}\{v_i\}$, $i = 1,2,3$. Take $C = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$, where $\alpha_1, \alpha_2, \alpha_3 \geqslant 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. If two of $\alpha_1, \alpha_2, \alpha_3$ are zero then clearly $i(C) = (n-1,1,0)$. If at least two of $\alpha_1, \alpha_2, \alpha_3$ are positive, then the preceding observation shows that $u^*Cu > 0$ for all nonzero u with $u_1 = 0$. It follows that C has at least $n-1$ positive eigenvalues. As the intersection $E_1^0 \cap E_2^0 \cap E_3^0$ is nonempty, a vector $w \in \mathbb{R}^n$ exists such that $w^*A_iw < 0$, $i = 1,2,3$. Then $w^*Cw < 0$, and the inertia of C is $(n-1,1,0)$.

If A_1, A_2, A_3 were simultaneously inertia explicable, we could find a hyperplane K in $\mathbb{P}^{n-1}$ which did not meet any of E_1, E_2, or E_3. Choose a point Q_4 in $E_1^0 \cap E_2^0 \cap E_3^0$, and choose it so that the 2-dimensional plane M determined by Q_1, Q_2, and Q_4 meets K in a line L not belonging to the hyperplane at infinity.

Now $M \cap \partial E_i$ is an ellipse tangent to the line $M \cap H$ at Q_i, $i = 1,2$. Moreover, $M \cap \partial E_3$ is a hyperbola with one sheet tangent to $M \cap H$ at Q_3. It follows that the intersection of M with the three conics E_1, E_2, E_3 gives a picture as indicated in Figure 1. Now the line L belongs to K, and thus does not meet any of the conics above. But if L does not intersect the triangle $Q_1Q_2Q_4$, it meets the hyperbola, and if L meets $Q_1Q_2Q_4$, it meets at least one ellipse. This proves Lemma 7 and verifies Theorem 4. ∎

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